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p-forms on d-spherical tessellations

J.S. Dowker

Theory Group, School of Physics and Astronomy, The University of Manchester, Manchester, England, United Kingdom

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Abstract

The spectral properties of p-forms on the fundamental domains of regular tessellations of the d-dimensional sphere are discussed. The degeneracies for all ranks, p, are organised into a double Poincaré series which is explicitly determined. In the particular case of coexact forms of rank $(d-1)/2$, for odd d, it is shown that the heat-kernel expansion terminates with the constant term, which equals $(-1)^{p+1}/2$, and that the boundary terms also vanish, all as expected. As an example of the double-domain construction, it is shown that the degeneracies on the sphere are given by adding the absolute and relative degeneracies on the hemisphere, again as anticipated. The eta invariant on S^3/Γ is computed to be irrational.

The spectral counting function, $N(\lambda)$, is calculated and the accumulated degeneracy given exactly. The Weyl conjecture is shown not to be valid for the exact $N(\lambda)$ but does hold for an averaged form, $\overline{N}(\lambda)$. A generalised Weyl–Polya conjecture for p-forms is suggested and verified and the exact Polya conjecture is tested numerically on the hemisphere.

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1. Introduction

In a recent work, [1], I have looked at p-forms on tessellations of the 3-sphere. In this follow-up, I expand on the higher dimensional aspects of the formalism initiated there. The generating functions for the degeneracies of the de Rham Laplacian eigenforms are presented for any p-form, although I concentrate, for actual ζ-function computations, on (coexact) forms of the middle rank, $p = (d-1)/2$, on the odd d-dimensional factored sphere, S^d/Γ for arbitrary d. The reason for this is that the eigenvalues are perfect squares and the expressions for the spectral objects can be taken a long way in terms of known quantities.

The deck group, Γ, is the complete symmetry group of a regular polytope in $n, = d+1$, dimensions. In another terminology, it is a real reflection group. These have all been classified.

This paper should be looked upon as a direct continuation of [1] and, to avoid repetition, I will use, without derivation, any necessary equations and results of this reference. As there, the analysis is presented as an example of

E-mail address: dowker@man.ac.uk.

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spectral theory in bounded domains that is easily, and explicitly, managed *via* images. It largely consists of bolting together already existing pieces of knowledge and taking the expressions somewhat further than seems to have been done in the literature. This leads to very compact expressions for the generating functions.

The spectral counting function will also be calculated, allowing some aspects of the Weyl and Polya conjectures to be examined.

General comments on physical motivation were given in [1] related mainly to speculations on the topology of the universe and applications to string theory. Additionally, the quantum field theory of anti-symmetric fields has a certain importance, see e.g. [2–5], and I will consider a selection of spectral objects, such as heat-kernel expansion coefficients, the Casimir energy and the eta invariant, as standard examples of frequent interest.

The middle rank form theory is conformally invariant and could be considered as a higher dimensional Maxwell theory. In this special case, the degeneracy generating function is, up to a trivial factor, the one-particle thermal partition function from which the Fock space, field theory free energy can be found. This aspect will not be explored in this essentially mathematical paper.

2. Spectrum and generating functions

The coexact eigenvalues of the Hodge–de Rham Laplacian, $d\delta + \delta d$, on the d-sphere are standard,

$$\lambda^{CE}(p,l) = (l + (d+1)/2)^2 - ((d-1)/2 - p)^2, \quad l = 0, 1, \ldots, \tag{1}$$

which specialises to

$$\mu^{CE}(p,l) = (l + p + 1)^2, \quad l = 0, 1, \ldots, \tag{2}$$

for middle rank forms, if d is odd.

The corresponding degeneracies, $g_b^{CE}(p,l)$, are best encoded in the generating function defined by

$$g_b^{CE}(p,\sigma) \equiv \sum_{l=0}^{\infty} g_b^{CE}(p,l)\sigma^l, \tag{3}$$

where the label, $b = r$ or a, indicates the conditions satisfied by the p-form on the boundary, $\partial\mathcal{M}$, of the fundamental domain, $\mathcal{M}$, for the action of Γ on S^d. Absolute (‘a’) conditions arise when the form is symmetric under this action while relative (‘r’) ones originate from anti-symmetric behaviour. The relation is the duality one,

$$g_b^{CE}(p,\sigma) = g_{*b}^{CE}(d-1-p,\sigma), \tag{4}$$

which is a consequence of the $\mathbb{R}^n$ duality,

$$h_{*b}^{CCC}(n-p,\sigma) = h_b^{CCC}(p,\sigma), \tag{5}$$

for the closed–coclosed functions, h^{CCC}, and the relation between coexact and closed,

$$g_b^{CE}(p,\sigma) = h_b^{CCC}(p+1,\sigma). \tag{6}$$

An important fact is that forms of the middle rank are self-dual in the sense that $g_b^{CE}((d-1)/2,\sigma) = g_{*b}^{CE}((d-1)/2,\sigma)$.

Equations are developed in [1] that allow the generating function to be found in closed form in terms of the degrees, $\mathbf{d} = (d_1, d_2, \ldots, d_d)$, that define the polytope (reflection) group, Γ. The case of $p = 1$, $d = 3$ was treated in detail there. Now I expose the general result for *any* coexact p-form in *any* dimension d,

$$g_a^{CE}(p,\sigma) = (-1)^{d+1+p} \frac{\sum_{q=0}^{d-1-p} (-1)^q e_{d-q}(\sigma^{d_1}, \ldots, \sigma^{d_d})}{\sigma^{p+1} \prod_{i=1}^{d} (1 - \sigma^{d_i})}, \tag{7}$$

where the e_q are the elementary symmetric functions.

It is useful to write out the relative generating function from the duality relation (4),

$$g_r^{CE}(p,\sigma) = (-1)^p \frac{\sum_{q=0}^{p}(-1)^q e_{d-q}(\sigma^{d_1},\ldots,\sigma^{d_d})}{\sigma^{d-p}\prod_{i=1}^{d}(1-\sigma^{d_i})},$$

$$= (-1)^{p+d} \frac{\sum_{q=d-p}^{d}(-1)^q e_{q}(\sigma^{d_1},\ldots,\sigma^{d_d})}{\sigma^{d-p}\prod_{i=1}^{d}(1-\sigma^{d_i})}. \tag{8}$$

I derive these expressions later, while developing the formalism.

As in [1], the behaviour under the inversion $\sigma \to 1/\sigma$ is important. From (7),

$$g_a^{CE}(p,1/\sigma) = (-1)^{p+1}\sigma^{p+1} \frac{\sum_{q=0}^{d-1-p}(-1)^q e_{q}(\sigma^{d_1},\ldots,\sigma^{d_d})}{\prod_{i=1}^{d}(1-\sigma^{d_i})}, \tag{9}$$

and combined with (8), this gives

$$T_a^{CE}(p,1/\sigma) - (-1)^d T_r^{CE}(p,\sigma) = \sigma^{p-(d-1)/2}(-1)^{p+1}, \tag{10}$$

after defining

$$T_b^{CE}(p,\sigma) = \sigma^{(d+1)/2} g_b^{CE}(p,\sigma). \tag{11}$$

For self-dual forms, d is odd and (10) gives the *symmetrical* part of the 'cylinder kernel',

$$T_b^{CE}(p,1/\sigma) + T_b^{CE}(p,\sigma) = (-1)^{p+1}, \quad (b=a,r), \tag{12}$$

which can be employed to advantage when evaluating the ζ-function in the next section.

3. Zeta functions and heat kernels

A useful spectral organising quantity is the coexact ζ-function,

$$\zeta_b^{CE}(p,s) = \sum_{l=0}^{\infty} \frac{g_b^{CE}(p,l)}{\lambda^{CE}(p,l)^s},$$

because on a manifold, with or without a boundary, there is the general decomposition

$$\zeta_b(p,s) = \zeta_b^{CE}(p,s) + \zeta_b^{CE}(p-1,s), \quad p>0, \tag{13}$$

of the total ζ-function for the Hodge–de Rham Laplacian.

Only for middle rank forms can ζ^{CE} be related, using (2), to the generating functions and, for the remainder of this section, I will make this simplifying restriction so that $d=2p+1$. The ζ-function is then

$$\zeta_a^{CE}(p,s) = \frac{\mathrm{i}\Gamma(1-2s)}{2\pi}\int_{C_0} \mathrm{d}\tau(-\tau)^{2s-1} T_a^{CE}(p,\tau)$$

$$= \frac{\mathrm{i}\Gamma(1-2s)}{2\pi}\int_{C_0} \mathrm{d}\tau(-\tau)^{2s-1} \mathrm{e}^{-\tau(d+1)/2} g_a^{CE}(p,\tau) \tag{14}$$

where I have introduced $\tau = -\log\sigma$ and understand, notationally, $T_a^{CE}(p,\tau) \equiv T_a^{CE}(p,\sigma)$ and $g_a^{CE}(p,\tau) \equiv g_a^{CE}(p,\sigma)$. (I have written the absolute quantity, but this equals the relative one here.)

Therefore, from (7),

$$\zeta_a^{CE}(p,s) = (-1)^p \frac{i\Gamma(1-2s)}{2\pi}\int_{C_0} d\tau (-\tau)^{2s-1}\sum_{q=0}^{d-1-p}(-1)^q \frac{e_{d-q}(e^{-d_1\tau},\ldots,e^{-d_d\tau})}{\prod_{i=1}^{d}(1-e^{-d_i\tau})}. \tag{15}$$

I now recall the integral representation of the Barnes ζ-function,

$$\zeta_d(s,a|\mathbf{d}) = \frac{i\Gamma(1-s)}{2\pi}\int_{C_0} d\tau \frac{e^{-a\tau}(-\tau)^{s-1}}{\prod_{i=1}^{d}(1-e^{-d_i\tau})}, \tag{16}$$

so that (15) becomes

$$\begin{aligned}\zeta_a^{CE}(p,s) &= (-1)^p \sum_{q=0}^{d-1-p}(-1)^q \sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d} \zeta_d(2s, \Sigma d - d_{i_1} - \cdots - d_{i_q}|\mathbf{d})\\ &= (-1)^p \sum_{q=0}^{d-1-p}(-1)^q \sum_{\substack{i_1<i_2<\cdots<i_{d-q}\\=1}}^{d} \zeta_d(2s, d_{i_1}+\cdots+d_{i_{d-q}}|\mathbf{d})\\ &= (-1)^{p+d} \sum_{q=d-p}^{d}(-1)^q \sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d} \zeta_d(2s, d_{i_1}+\cdots+d_{i_q}|\mathbf{d}),\end{aligned} \tag{17}$$

by a simple reordering.

As an example, I calculate the values $\zeta_a^{CE}(p,-k/2)$, $k\in\mathbb{Z}$. Averaging the first and third lines of (17),

$$\begin{aligned}\zeta_a^{CE}(p,-k/2) &= (-1)^p \frac{k!}{2(d+k)!\prod d_i}\left(\sum_{q=0}^{d-1-p}(-1)^{q+k} + \sum_{q=d-p}^{d}(-1)^q\right)\\ &\quad\times \sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d} B^{(d)}_{d+k}(d_{i_1}+\cdots+d_{i_q}|\mathbf{d}).\end{aligned}$$

If k is even, the two sums combine,

$$\zeta_a^{CE}(p,-k) = \tfrac{(-1)^p(2k)!}{2(d+2k)!\prod d_i}\sum_{q=0}^{d}(-1)^q \sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d} B^{(d)}_{d+2k}(d_{i_1}+\cdots+d_{i_q}|\mathbf{d}). \tag{18}$$

Special interest is attached to the value $k=0$,

$$\zeta_a^{CE}(p,0) = \tfrac{(-1)^p}{2d!\prod d_i}\sum_{q=0}^{d}(-1)^q \sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d} B^{(d)}_{d}(d_{i_1}+\cdots+d_{i_q}|\mathbf{d}). \tag{19}$$

To evaluate these, I note that the Barnes ζ-function satisfies the recursion, [6],

$$\zeta_d(s,a+d_i|\mathbf{d}) = \zeta_d(s,a|\mathbf{d}) - \zeta_{d-1}(s,a|\hat{\mathbf{d}}_i),$$

whose limiting iteration is, [7],

$$\begin{aligned}&\zeta_d(s,a+d_1+\cdots+d_d|\mathbf{d}) - \sum_{*=1}^{d}\zeta_d(s,a+d_1+\cdots+*+\cdots+d_d|\mathbf{d})\\ &\quad+\sum_{*=1}^{d}\sum_{*=1}^{d}\zeta_d(s,a+d_1+\cdots+*+\cdots+*+\cdots+d_d|\mathbf{d})\end{aligned}$$

$$\vdots$$
$$+(-1)^{d-1}\sum_{i=1}^{d}\zeta_d(s,a+d_i|\mathbf{d})+(-1)^d\zeta_d(s,a|\mathbf{d})=a^{-s}. \tag{20}$$

In the first summation, the star denotes that one of the d's is to be omitted, in turn. In the second summation, every two different pairs of d's must be successively omitted, and so on. This is the notation of Barnes [6]. The star summations and omissions correspond to the more conventional ordered summations in e.g. (17).

The iteration, (20), on setting s to specific values or by extracting the poles of the ζ-function, leads to identities involving the generalised Bernoulli polynomials. For example, setting s equal to $-2k$, it is possible to safely equate a to zero and (20) gives the values of the expressions (18) and (19). I find

$$\begin{aligned}\zeta_a^{CE}(p,-k)&=0\\ \zeta_a^{CE}(p,0)&=\frac{1}{2}(-1)^{p+1},\end{aligned} \tag{21}$$

which show that the coexact middle form heat-kernel coefficients, $C^{CE}_{k+d/2}$, $k=1,2,\ldots$, vanish and that the constant term, $C^{CE}_{d/2}$, equals $(-1)^{p+1}/2$. This derivation is related to, but independent of, that presented in [1]. It does not depend on the fact that the $C_{d/2}(p)$ coefficient for a general p-form vanishes on the *doubled* fundamental domain,

$$C^b_{d/2}(p)+C^{*b}_{d/2}(p)=0,\quad \forall p, \tag{22}$$

and could be taken as a proof of this fact.

Furthermore, as mentioned at the end of the previous section, the symmetrical part of the integrand, (12), produces (21) immediately, bypassing explicit use of the recursion formulae which I have given, though, for completeness.

The heat-kernel expansion terminates with the constant term, which generalises the result in [8] on the full sphere. It is almost obvious that factoring the sphere will not alter this fact. The only question would be the effect of the fixed points.

Adding the two (equal) constants for absolute and relative conditions gives the constant for the doubled fundamental domain. In particular, the value, $(-1)^{p+1}$, holds for the full sphere. This agrees with the known value, [8–10].

The remaining heat-kernel coefficients, $C^{CE}_{k/2}$, follow from the 'positive' poles of the coexact ζ-function at $s=(d-k)/2$, for $k=0,1,\ldots,d-1$, which themselves result from the known poles of the Barnes ζ-function according to (17),

$$C^{CE}_{k/2}=(-1)^p\frac{\Gamma((d-k)/2)}{2k!(d-k-1)!\prod d_i}\left(\sum_{q=0}^{d-1-p}(-1)^q-\sum_{q=d-p}^{d}(-1)^{q+k}\right)\sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d}B_k^{(d)}(d_{i_1}+\cdots+d_{i_q}|\mathbf{d}).$$

If k is odd, the summations combine to allow the pole part of (20) to come into use showing that the boundary coefficients, i.e. $C^{CE}_{k/2}$ for k odd, are zero, again generalising the result in [1]. As before, this conclusion follows more easily from (12).

There are no such 'topological' simplifications or cancellations for the other values of the ζ-function, for example for the coexact Casimir energy,

$$\begin{aligned}E&=\frac{1}{2}\zeta_a^{CE}(p,-1/2)\\ &=\frac{(-1)^p}{2(d+1)!\prod d_i}\sum_{q=0}^{p}(-1)^q\sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d}B^{(d)}_{d+1}(d_{i_1}+\cdots+d_{i_q}|\mathbf{d}),\end{aligned}$$

and one is reduced to actual computation.

4. Extensions and elaborations

Although the expression, (14), for $\zeta_b^{CE}(p,s)$ is valid just for the middle rank forms on S^d/Γ, it has significance for *all* p when the factored sphere is realised as the base of a generalised cone in $\mathbb{R}^{d+1}$, [11,12]. I have also referred to this construction as a bounded Möbius corner, [1,13]. The separation of variables into radial and angular introduces a term that effectively cancels the second part of (1). Gilkey, [14] Section 4.7.5, refers to the resulting operator as the *normalised* spherical Laplacian.

In this case the T quantities defined in (11) are *bone fide* cylinder kernels (without propagation significance) and the corresponding absolute ζ-function is

$$\begin{aligned}
\zeta_a^{CE}(p,s) &= \frac{\mathrm{i}\Gamma(1-2s)}{2\pi}\int_{C_0} \mathrm{d}\tau(-\tau)^{2s-1}T_a^{CE}(p,\tau)\\
&= (-1)^p\frac{\mathrm{i}\Gamma(1-2s)}{2\pi}\int_{C_0}\mathrm{d}\tau(-\tau)^{2s-1}\mathrm{e}^{-((d-1)/2-p)\tau}\sum_{q=0}^{d-1-p}(-1)^q\frac{e_{d-q}(\mathrm{e}^{-d_1\tau},\ldots,\mathrm{e}^{-d_d\tau})}{\prod_{i=1}^{d}(1-\mathrm{e}^{-d_i\tau})}\\
&= (-1)^d\sum_{q=p+1}^{d}(-1)^q\sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d}\zeta_d\left(2s,\frac{d-1}{2}-p+d_{i_1}+\cdots+d_{i_q}|\mathbf{d}\right),
\end{aligned}\tag{23}$$

with duality giving the relative $\zeta_r(p,s)=\zeta_a(d-1-p,s)$,

$$\zeta_r^{CE}(p,s)=(-1)^d\sum_{q=d-p}^{d}(-1)^q\sum_{\substack{i_1<i_2<\cdots<i_q\\=1}}^{d}\zeta_d\left(2s,p-\frac{d-1}{2}+d_{i_1}+\cdots+d_{i_q}|\mathbf{d}\right).\tag{24}$$

These ζ-functions appear as useful intermediate quantities but have no independent dynamical significance. In our work on the ball, Dowker and Kirsten [11], they were referred to as 'modified' ζ-functions. The present results would allow us to extend the ball calculations to factored bases in a systematic fashion. For example the computations of the scalar functional determinants reported in [13] could be generalised to p-forms.

An expression for the modified ζ-function on the full sphere is given in Eq. (42) in [11]. A related formula can be obtained from our present results by adding the absolute and relative expressions on the hemisphere, for which all the degrees are one. Hence from (23) and (24),

$$\begin{aligned}
\zeta_{\text{sphere}}^{CE}(p,s) = (-1)^d\Bigg(&\sum_{q=p+1}^{d}(-1)^q\binom{d}{q}\zeta_d\left(2s,\frac{d-1}{2}-p+q|\mathbf{d}\right)\\
&+\sum_{q=d-p}^{d}(-1)^q\binom{d}{q}\zeta_d\left(2s,p-\frac{d-1}{2}+q|\mathbf{d}\right)\Bigg),
\end{aligned}$$

which, after some manipulation, is equivalent to the form given in [11].

More generally, adding relative and absolute produces the results for a 'doubled' fundamental domain. Rather than give the full expressions, I will only look at the consequences of the duality relation, (10), which yields

$$T_{a+r}^{CE}(p,\tau)-(-1)^d T_{a+r}^{CE}(p,-\tau)=2(-1)^{p+d}\cosh(p-(d-1)/2)\tau,$$

allowing ζ-function values to be easily found as powers of $p-(d-1)/2$. I will not do this in detail and only remark that these values are independent of the factoring, Γ.

5. The eta invariant

An important spectral quantity is the eta invariant, $\eta(0)$, which gives a measure of the asymmetry of the spectrum. Originally introduced by Atiyah, Patodi and Singer, [15], as a boundary 'correction' to an index, it has achieved an independent life, and its computation has become a standard challenge. A number of approaches, simple and sophisticated, are available. The original one, [15], employs the G-index theorem. According to Donnelly, [16], Millson was the first to evaluate $\eta(0)$ on lens spaces by direct calculation. A direct spectral computation, in the particular case of spherical space forms, is also mentioned in [15] and attributed to Ray. Such a calculation was given, later, by Katase, [17], on quotients of the 3-sphere. The analysis is somewhat involved and the result is just the general angle form given previously. Even so, I outline my own version below.

The actual numbers for the various homogeneous (fixed point free) quotients of S^3 were computed by Gibbons et al., [18], who performed the group average by summing over the angles that define the elements. Something similar was done by Seade [19]. In [20], I offered an algebraic alternative to this rather cumbersome geometric technique. The eta invariant on spherical space forms has been systematically investigated by Gilkey [14,21].

The eta function, $\eta(s)$, measures the asymmetry of the boundary part of an operator and, as such, is computed on a closed manifold. As an exercise, I wish to find it for fundamental domains associated with the quotient S^d/Γ and these have a boundary. In fact, things are not quite so bad because I can work on the doubled fundamental domains resulting from the restriction to the direct rotational polytope group. However, the domain still has edges and vertices.

The fundamental domain, $\mathcal{M}$, of the spherical tessellation is the base of the generalised (metric) cone formed, on $\mathbb{R}^n$, by the set of reflecting hyperplanes that define the extended group Γ. As such, it is part of the boundary of this cone, or Möbius corner (kaleidoscope), the other part being the union of the flat sides. Restricting to the rotational subgroup of Γ turns the cone into a periodic one whose boundary is just its base, the doubled fundamental domain, $2\mathcal{M}$. There are, however, singularities of codimension 2, corresponding to the edges of the fundamental domain, and of codimension 3 from the vertices; cf. [22].

The signature eta function on a d-manifold, N (typically a boundary), is neatly expressed in terms of the middle rank coexact eigenforms, ϕ_l, by, [23],

$$\eta(s) = \sum_l \int_N \frac{\phi_l^* \wedge \mathrm{d}\phi_l}{\mu_l^{s+1/2}},$$

where $\mu_l = \mu^{CE}(p, l)$ of (2), and $p = (d-1)/2$ is odd, ($d = 4D - 1$). For each label, l, there are, possibly, two coexact eigenforms ('positive' and 'negative') that can be chosen[1] to be eigenforms of $*d$,

$$*d\phi_l = \pm\omega_l\phi_l, \quad \omega_l = \sqrt{\mu_l},$$

and the sum is over both types. The ω spectrum is not generally symmetric.

Despite my preference for the algebraic method, I initially use angle summation since all the hard work has been done on the 3-sphere. Only the signature eta function will be considered and I now derive, again, the expression obtained long ago in [15].

In physicist's language, the signature eta function in four dimensions is just the transverse spin-1 spectral asymmetry function on the three-dimensional boundary, [24,25]. For spherical factors, the necessary spectral information has been given a number of times before in various connections, e.g. [26,27], and repeated in [20,28].

In terms of the positive and negative, spin-1 'Hamiltonian' ζ-functions, on S^3/Γ,

$$\begin{aligned} \zeta^+(s) &= \sum_{\overline{L}=1}^{\infty} \frac{d^+(\overline{L})}{(\overline{L}+1)^s} \\ \zeta^-(s) &= \sum_{\overline{L}=3}^{\infty} \frac{d^-(\overline{L})}{(\overline{L}-1)^s} = \sum_{\overline{L}=1}^{\infty} \frac{d^-(\overline{L}+2)}{(\overline{L}+1)^s}, \end{aligned} \tag{25}$$

the eta function is defined to be

$$\eta(s) = \zeta^+(s) - \zeta^-(s).$$

The degeneracies, $d^\pm$, follow from character theory and are given in the just cited references. The eta function can be written as the group average,

[1] The restriction $d = 4D - 1$ is necessary for this. I have also allowed for complex eigenforms, although they can be arranged to be real. I have not distinguished notationally between the positive and negative types.

$$\eta(s) = \frac{1}{|\Gamma|}\sum_{\gamma}\eta(\gamma, s), \tag{26}$$

where, by the algebra detailed in [28], the partial eta function, $\eta(\gamma, s)$, is

$$\eta(\gamma, s) = 2\sum_{n=1}^{\infty}\frac{1}{n^s}\frac{\sin\alpha\sin n\beta - \sin\beta\sin n\alpha}{\cos\alpha - \cos\beta}. \tag{27}$$

The sum over γ in (26) is a sum over the angles, α and β.

The important value is $\eta(0)$ and substitution into (27) shows that there are no problems with the fixed points. As usual, the identity element gives zero as do the other special values, $\alpha = 0$, $\beta \neq 0$, which correspond to the fixing of a 2-flat in the ambient $\mathbb{R}^4$. The summation over n is then trivially performed using $2\sum_{n=1}^{\infty}\sin n\theta = \cot\theta/2$ giving

$$\eta(0) = -\frac{1}{|\Gamma|}\sum_{\alpha\neq 0;\beta}\cot\alpha/2\cot\beta/2, \tag{28}$$

which is, apart from the summation restriction, the standard formula.[2]

The values of α and β corresponding to the elements of the several polytope groups can now be inserted and the group average performed using the class decompositions given in [1] which were taken from Hurley [29] and Chang [30]. I find the values for the doubled fundamental domain,

$$\begin{aligned}
\eta(0) &= -\frac{2}{5\sqrt{5}}, \quad \{3^3\}\\
&= -\frac{5}{16}, \quad \{3^2 4\}\\
&= -\frac{29}{48}, \quad \{343\}\\
&= -\frac{2341}{5400} - \frac{118}{75\sqrt{5}}, \quad \{3^2 5\}.
\end{aligned}$$

The novelty is the presence of the surd in two cases and the simple fractions in the others. By contrast, in the evaluation of the Casimir energy all irrationalities cancel.

The eta function on a lune is, almost trivially, zero because all group elements fix a 2-flat 'axis' of rotation, and these contribute nothing.

Without going through the mode analysis, it is reasonable that the Dirac eta invariant will be given by the standard, basic expression as given in Hanson and Römer, [31], e.g., see [32],

$$\eta_S(0) = -\frac{1}{2|\Gamma|}\sum_{\alpha\neq 0;\beta}\operatorname{cosec}\alpha/2\operatorname{cosec}\beta/2. \tag{29}$$

Numerical evaluation yields the following values:

$$\begin{aligned}
\eta_S(0) &= -\frac{1}{5\sqrt{5}}, \quad \{3^3\}\\
&= -\frac{89}{768} - \frac{9}{32\sqrt{2}}, \quad \{3^2 4\}\\
&= -\frac{1867}{1728} - \frac{9}{8\sqrt{2}}, \quad \{343\}\\
&= -\frac{37\,291}{7200} + \frac{277}{75}\frac{1}{\sqrt{5}}, \quad \{3^2 5\}.
\end{aligned}$$

[2] This derivation is, no doubt, the same as those of Millson and Ray mentioned earlier.

The presence of the surds implies that it is not possible to find alternative expressions for the eta invariant purely in terms of the degrees, d_i, as it is for *homogeneous*, fixed point free quotients, [20], or for the Casimir energy.

Incidentally, this conclusion seems not in agreement with the work of Degeratu, [33], which relates the coefficients in the Laurent expansion of the Molien (Poincaré) series directly to the (Dirac) eta invariants associated with the boundaries, S^3/Γ_i, of the orbifolds, $\mathbb{C}^2/\Gamma_i$ where the Γ_i are subgroups of Γ.

Cheeger, [23], discusses the eta invariant on a generalised cone. For the standard situation of a smooth manifold, M, a generalised cone is attached to the boundary, N, converting M into X, a compact space, with a conical singularity, on which the index can be calculated. In this way, Cheeger shows that the standard Atiyah–Patodi–Singer formula for $\mathrm{Sig}(M)$ follows from spectral analysis on the cone, the boundary $\eta(0)$ arising now from the effect of the cone apex. Also mentioned is the non-standard eta function on manifolds with boundaries or with conical points and the possibility that it might be irrational is raised. My computation seems to confirm this and I leave it at this point.

Dirac η functions on manifolds with boundary were discussed by Douglas and Wojciechowski [34]. See also Dai, [35].

6. Developing the formalism — the double Poincaré series

In this section I present a derivation of my basic formulae, (7) and (8), from the recursions for the various generating functions given in [1] which are defined as sums over the mode label, as in (3). Although not necessary, I will do this using double generating functions obtained from the previous ones by summing also over the form rank, p. Such double series are used by Ray on spheres, [36], but my approach is different in detail and, in fact, refers to the expressions *after* the group average. They allow for a compressed treatment.

I start with the degeneracy of harmonic polynomial forms on $\mathbb{R}^n$ and define the double Poincaré series, a finite, ‘fermionic’ polynomial in z,

$$h_b(z,\sigma)=\sum_{p=0}^{n}h(p,\sigma)z^p=\frac{1-\sigma^2}{|\Gamma|}\sum_A\frac{\det(1+zA)}{\det(1-\sigma A)}\chi^*(A). \tag{30}$$

It is possible to think of z as a fugacity.

Using standard identities, explicit forms are

$$h_a(z,\sigma)=(1+z\sigma)\prod_{i=1}^{d}\frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}}, \tag{31}$$

and

$$\begin{aligned}h_r(z,\sigma)&=(z+\sigma)\prod_{i=1}^{d}\frac{z+\sigma^{m_i}}{1-\sigma^{d_i}}\\&=z^nh_a\left(\frac{1}{z},\sigma\right),\end{aligned} \tag{32}$$

where the final factor in the numerator has been extracted using $m_n=1$. Eq. (30) without the harmonic factor $(1-\sigma^2)$ is Solomon’s theorem, [37], Bourbaki, [38], p. 136, Kane, [39], Section 22.4. See also, relatedly, Sturmfels, [40], p. 37, Exercise 5. The explicit forms are algebraic while the group average is geometric.

I introduce the notion of polynomial dual, or reciprocal, by the definition

$$^*f(z)\equiv z^nf(1/z),\quad \text{i.e. } ^{**}f(z)=f(z),$$

on a polynomial, f, of (unwritten) degree n, so that, e.g., the relation (32) reads

$$h_{*b}(z,\sigma)={}^*h_b(z,\sigma). \tag{33}$$

This helps notationally when dealing with the maximum form rank, $p=n$, on $\mathbb{R}^n$.

The recursions given in [1] transcribe into formulae that can be solved algebraically. I give the basic steps. For example, the harmonic and closed harmonic generating functions are related by the recursion

$$h_b(p,\sigma) = h_b^C(p,\sigma) + \sigma h_b^C(p+1,\sigma) \tag{34}$$

which, on account of $h_b^C(p,\sigma) = 0$ $(p > n)$, becomes, using the same basic symbols,

$$h_b(z,\sigma) = \overline{h}_b^C(z,\sigma) + \frac{\sigma}{z}(\overline{h}_b^C(z,\sigma) - \overline{h}_b^C(0,\sigma)), \tag{35}$$

where $\overline{h}$ is defined by

$$\overline{h}_b^C(z,\sigma) = h_b^C(z,\sigma) - \sigma^2\delta_{ba}. \tag{36}$$

In going from (34) to (35), the term involving σ^2 has been inserted by hand, *via* (36), for the reason mentioned in [1] for adding a term, $\delta_{ba}\delta_{p0}\delta_{l2}$, to the solution of the recursion for $h_b^C(p,l)$. It is needed to ensure the required zero-mode end point value, $h_b^C(0,\sigma) = \delta_{ba}$, which is not covered by the exact sequence that provides the recursion.

It is useful at this point to list some of the end point values,

$$\begin{aligned} &h_b^C(0,\sigma) = \delta_{ba}, \qquad {}^*h_b^C(0,\sigma) = {}^*h_b(0,\sigma), \qquad h_b^{CCC}(0,\sigma) = \delta_{ba}, \\ &{}^*h_b^{CCC}(0,\sigma) = \delta_{br}, \qquad {}^*h_b^{CC}(0,\sigma) = \delta_{br}, \qquad h_b^{CC}(0,\sigma) = h_b(0,\sigma). \end{aligned} \tag{37}$$

The identities established in [1] also take on an elegant appearance. For example the supertraces,

$$h_b(-\sigma,\sigma) = \delta_{ba}(1-\sigma^2), \qquad h_b^{CC}(-\sigma,\sigma) = \delta_{ba}. \tag{38}$$

The first equation is actually an easy consequence of (30).

Thus, setting $z = -\sigma$ in (35) gives

$$h_b(-\sigma,\sigma) = \overline{h}_b^C(0,\sigma) = (1-\sigma^2)\delta_{ba},$$

as an algebraic check.

The solution of (35) is

$$h_b^C(z,\sigma) = \frac{z}{z+\sigma}h_b(z,\sigma) + \frac{1+z\sigma}{1+z/\sigma}\delta_{ba}. \tag{39}$$

For the closed and the closed–coclosed functions there are no ‘correction terms’ and the recursion formula, which has the same form as (35), gives

$$h_b^{CCC}(z,\sigma) = \frac{z}{z+\sigma}h_b^{CC}(z,\sigma) + \frac{\sigma}{z+\sigma}\delta_{ba}, \tag{40}$$

using $h_b^{CCC}(0,\sigma) = \delta_{ba}$.

The left-hand side of (40) is the quantity required as it encodes the closed degeneracies on the d-sphere, [41]. The inputs are the explicit forms (31) and (32) which can be used after relating h^{CC} and h^C by duality on $\mathbb{R}^n$. This yields the various equivalent forms,

$$\begin{aligned} h_b^{CC}(z,\sigma) &= z^n h_{*b}^C\left(\frac{1}{z},\sigma\right) = {}^*h_{*b}^C(z,\sigma) \\ &= \frac{z^n}{1+z\sigma}h_{*b}\left(\frac{1}{z},\sigma\right) + \frac{z^n\sigma(z+\sigma)}{1+z\sigma}\delta_{*ba} \\ &= \frac{1}{1+z\sigma}h_b(z,\sigma) + \frac{z^n\sigma(z+\sigma)}{1+z\sigma}\delta_{br} \end{aligned} \tag{41}$$

and

$${}^*h_b^{CC}(z,\sigma) = h_{*b}^C(z,\sigma) = \tfrac{z}{z+\sigma}h_{*b}(z,\sigma) + \tfrac{1+z\sigma}{1+z/\sigma}\delta_{br}, \tag{42}$$

using (39). These expressions exhibit immediately the end values in (37).

Substitution into (40) produces

$$h_b^{CCC}(z,\sigma)=\frac{1}{(1+\sigma/z)(1+\sigma z)}h_b(z,\sigma)+\frac{z^n z\sigma}{1+z\sigma}\delta_{br}+\frac{\sigma}{z+\sigma}\delta_{ba}. \tag{43}$$

For consistency, duality on $\mathbb{R}^n$ in the form

$$h_b^{CCC}(z,\sigma)={}^*h_{*b}^{CCC}(z,\sigma) \tag{44}$$

can easily be checked. Hence it is sufficient to restrict to absolute conditions and the explicit form (31) results in the final expression,

$$h_a^{CCC}(z,\sigma)=\frac{z}{z+\sigma}\prod_{i=1}^d\frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}}+\frac{\sigma}{z+\sigma}, \tag{45}$$

which is equivalent to (7), taking into account the relation (6).

One way of showing this is to go backwards and derive the recursion satisfied by the double Poincaré series constructed directly from (7), virtually paralleling the previous analysis. For simplicity, I rename $h_a^{CCC}(*,\sigma)$ as $H_a(*)$.

From (7) it is straightforward to derive the form rank recursion,

$$\sigma^{p+1}H_a(p+1)+\sigma^p H_a(p)=\frac{e_p(\sigma^{d_1},\dots,\sigma^{d_d})}{\prod_{i=1}^d(1-\sigma^{d_i})}. \tag{46}$$

I next note that, because of the end point value ${}^*H(0)=0$, the top limit in the sum defining $H(z)$ can be put at $p=d$ (appropriate for working on S^d) and the construction of the generating function of z allows the recursion (46) to be rewritten and then solved, much as before, to give

$$H_a(\sigma z)=\frac{z}{1+z}\prod_{i=1}^d\frac{1+z\sigma^{d_i}}{1-\sigma^{d_i}}+\frac{1}{1+z},$$

which is the same as (45) after setting $z\to z/\sigma$; moreover, one can see how the exponents, m_i, turn into the degrees, d_i. In this slightly synthetic way I have justified the forms (7) and (8). A direct demonstration is possible.

For convenience I give the expression for the absolute coexact double generating function that follows from (45) and the relation (6),

$$g_a^{CE}(z,\sigma)=\frac{1}{z+\sigma}\left[\prod_{i=1}^d\frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}}-1\right], \tag{47}$$

a neat encapsulation of my results. For completeness, the relative version reads[3]

$$\begin{aligned}g_r^{CE}(z,\sigma)&=\frac{1}{1+z\sigma}\left[\prod_{i=1}^d\frac{z+\sigma^{m_i}}{1-\sigma^{d_i}}+z^n\sigma\right]\\&=z^{d-1}\left(g_a^{CE}\left(\frac{1}{z},\sigma\right)+z\right).\end{aligned} \tag{48}$$

I remark again that the g^{CE} are polynomials in z.

On the d-hemisphere,

$$g_a^{CE}(z,\sigma)|_{\text{hemisphere}}=\frac{1}{z+\sigma}\left[\left(\frac{1+z}{1-\sigma}\right)^d-1\right], \tag{49}$$

[3] When checking the various relations it is necessary to note that, algebraically, $g_a^{CE}(p,\sigma)=1$ for $p=-1$.

and

$$z^d g_r^{CE}\left(\frac{1}{z},\sigma\right)|_{\text{hemisphere}} = \frac{z}{z+\sigma}\left[\left(\frac{1+z}{1-\sigma}\right)^d + \frac{\sigma}{z}\right], \tag{50}$$

while a supertrace result is

$$\sigma g_a^{CE}(-\sigma,\sigma) + d = \sum_{i=1}^{d}\frac{1}{1-\sigma^{d_i}}.$$

7. The 0-form case

When $z = 0$, i.e. $p = 0$, the above expressions for $g^{CE}(z,\sigma)$ do not give the degeneracies for the *general* 0-form. These are, of course, known, but the most convenient way of including them in the present formalism is to extend the range of l down to -1, which corresponds to a *zero mode*, as is apparent from (1). This mode exists only for absolute (i.e. Neumann) conditions and is a uniform function, a polynomial of zero order. Hence the *complete* 0-form generating function is

$$h_b(\sigma) = \delta_{ba} + \sigma g_b^{CE}(0,\sigma),$$

which, from (47) and (48), yield the known forms

$$h_a(\sigma) = h_N(\sigma) = \frac{1}{\prod_{i=1}^{d}(1-\sigma^{d_i})}$$

$$h_r(\sigma) = h_D(\sigma) = \frac{\sigma^{d_0}}{\prod_{i=1}^{d}(1-\sigma^{d_i})}, \quad d_0 = \sum_i m_i.$$

8. The double fundamental domain and a check

Formulae (47) and (48) give the generating functions on the fundamental domain, $\mathcal{M}$. That on the doubled domain, $2\mathcal{M}$, is obtained by adding these two expressions. This can be checked explicitly for the hemisphere since the full sphere degeneracies are standard, [41]. To this end, I split these as in [11],

$$g^{CE}(p,l) = \frac{(l+d)!}{p!(d-p-1)!l!}\left(\frac{1}{l+1+p} + \frac{1}{l+d-p}\right), \quad l = 0,1,\ldots, \tag{51}$$

the two parts of which are related by $p \to d-1-p$. Each part corresponds to a hemisphere contribution. Assuming, for a moment, that this is true, the combinatorial identity used in [11], Eq. (41), allows the eigenvalue generating functions on the hemisphere to be found from (51). Then,

$$g_a^{CE}(p,\sigma)|_{\text{hemisphere}} = \sum_{m=p+1}^{d}\binom{m-1}{p}\frac{1}{(1-\sigma)^m}$$

$$g_r^{CE}(p,\sigma)|_{\text{hemisphere}} = \sum_{m=d-p}^{d}\binom{m-1}{d-p-1}\frac{1}{(1-\sigma)^m}.$$

Construction of the double Poincaré series turns these into (49) and (50) by simple algebra (replace the lower limits by zero), confirming that the two parts do give the two hemisphere contributions. That is,

$$g^{CE}(z,\sigma)|_{\text{sphere}} = g_a^{CE}(z,\sigma)|_{\text{hemisphere}} + g_r^{CE}(z,\sigma)|_{\text{hemisphere}},$$

which is a special case of

$$g^{CE}(z,\sigma)|_{2\mathcal{M}} = g_a^{CE}(z,\sigma)|_{\mathcal{M}} + g_r^{CE}(z,\sigma)|_{\mathcal{M}},$$

a p-form generalisation of the known scalar result.

As well as the sum of relative and absolute expressions, the difference is also of interest (see the next section). It is preferable to remove the z^d term in (48) and define the combination

$$w(z,\sigma) = g_r(z,\sigma) - g_a(z,\sigma) - z^d \tag{52}$$

which is a polynomial of degree $d-1$ and is anti-reciprocal,

$$^*w(z,\sigma) = -w(z,\sigma). \tag{53}$$

9. The counting function

The counting function, $N(\lambda)$, could be considered the basic global spectral object.[4] In general terms, let the eigenvalues, λ_k, be ordered linearly $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$ with i a counting label and degeneracies accounted for by equality. I will use the definition of $N(\lambda)$,

$$N(\lambda) = \sum_{\lambda_k \le \lambda} 1. \tag{54}$$

If the spectrum is described by the distinct *eigenlevels* $\lambda(n)$, $n = 0, 1, 2, \ldots$, with explicit degeneracies $g(n) = g(\lambda(n))$, cf. Baltes and Hilf [42], then,

$$N(\lambda) = \sum_{\lambda(n) \le \lambda} g(n). \tag{55}$$

In the case under consideration in this paper, n is l, the polynomial order, and has a dynamical significance. The eigenvalues are functions of l, hence the notation.

The alternative definition, which is often used, e.g. [42],

$$N(\lambda) = \sum_{\lambda_k < \lambda} 1 + \sum_{\lambda_k = \lambda} \frac{1}{2}, \tag{56}$$

has the formal advantage that it can be written in the distributional form

$$N(\lambda) = \sum_{\lambda_k \le \lambda} \theta(\lambda - \lambda_k),$$

but here leads to slightly more involved algebra.

The value of $N(\lambda)$ depends on the particular function $\lambda(n)$ in the sense that, for a fixed argument, λ, it will vary if the form of $\lambda(n)$ is changed. One could, for example, add an arbitrary constant to the eigenvalues. On the other hand, the evaluation of $N(\lambda)$ at an eigenvalue, $G(n) \equiv N(\lambda(n))$, is the accumulated degeneracy and does not depend on the form of $\lambda(n)$,

$$G(n) = \sum_{n'=0}^{n} g(n').$$

$G(n)$ satisfies the recursion

$$G(n) - G(n-1) = g(n)$$

[4] Probably it first appears in Sturm's 1829 treatment of the roots of the secular equation.

which translates into the relation,

$$G(\sigma) = \frac{1}{1-\sigma} g(\sigma), \tag{57}$$

between generating functions, $G(\sigma) = \sum_{n=0}^{\infty} G(n)\sigma^n$ etc.

This equation can be applied to the situation in this paper and, for example, from (47) I find

$$G_a^{CE}(z,\sigma) = \frac{1}{(1-\sigma)(z+\sigma)} \left[\prod_{i=1}^{d} \frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}} - 1 \right]. \tag{58}$$

For the hemisphere, dividing out,

$$G_a^{CE}(z,\sigma) = \frac{1}{(1-\sigma)^2} \sum_{r=0}^{d-1} \left(\frac{1+z}{1-\sigma} \right)^r$$

and using the expansion

$$\sum_{l=0}^{\infty} \binom{l+r}{l} \sigma^l = \frac{1}{(1-\sigma)^{r+1}}$$

one readily deduces that, for each level,

$$G_a^{CE}(z,l) = \sum_{r=0}^{d-1} \binom{l+r+1}{l} (1+z)^r. \tag{59}$$

For a given eigenvalue form, such as (1), the counting function, $N(\lambda)$ can be determined by the condition that

$$\text{if} \quad \lambda(l) \le \lambda < \lambda(l+1) \quad \text{then } N(\lambda) = G(l). \tag{60}$$

Particular interest lies in the asymptotic behaviour of $N(\lambda)$ as $\lambda \to \infty$, in relation to Weyl's conjecture. It is tolerably clear that the leading term will follow from (59) as $l \to \infty$ since, from (1) $\lambda \sim l^2$ for very large λ. The highest power of l is, from (59),

$$G_a^{CE}(z,l) \sim G_{as}^{CE}(l) = \frac{l^d}{d!}(1+z)^{d-1}$$

which corresponds, after slight manipulation, to Weyl's term,

$$N_a(\lambda) \sim N_{as}(\lambda) = \binom{d-1}{p} \frac{|\mathcal{M}|}{(4\pi)^{d/2}} \frac{1}{\Gamma(1+d/2)} \lambda^{d/2}, \tag{61}$$

for *coexact* absolute p-forms when $\mathcal{M}$ is the hemisphere. The same result holds for relative conditions. With more algebraic work, these calculations can be extended to the general tessellation (58).

A point worth making here is that this leading term depends only on the degeneracies and not on the specific eigenvalues, so long as $\lambda = l^2 + O(l)$.

10. The Weyl conjecture

It is possible to go beyond this preliminary, crude analysis. As discussed in [43], the exact expressions allow one to investigate the validity of the Weyl conjecture which is generally taken to mean the statement that the exact counting function, $N(\lambda)$, for a finite volume d-manifold, $\mathcal{M}$, with boundary, asymptotically satisfies

$$N(\lambda) \sim A|\mathcal{M}|\lambda^{d/2} \pm B|\partial\mathcal{M}|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}), \tag{62}$$

where A and B are universal constants that depend only on the dimension. As Bérard, [44], says, the difficulty consists of showing the existence of an asymptotic expansion for $N(\lambda)$ and not in the explicit calculation of these constants, which, *given* the expansion, can easily be found using the heat-kernel expansion.

The conjecture is based on precise results for special cases, such as the rectangle, and the use of lattice theorems from number theory. The best result of this type is that of Kuznecov, [45], for those domains on which variables can be separated.

Weyl actually proved only that

$$N(\lambda) \sim A|\mathcal{M}|\lambda^{d/2} + o(\lambda^{d/2}). \tag{63}$$

The asymptotic behaviour (63) was proved for manifolds *with boundary* by Seeley, [46], and was refined by Melrose, [47], to the Weyl conjecture, (62), under special conditions:

(a) The boundary is (geodesically) concave and smooth.
(b) The set of periodic geodesics is of measure zero.
(c) No point on the boundary is fixed by the geodesic flow.

The second condition, introduced originally by Duistermaat and Guillemin, [48], is the vital one and is *necessary*.

So far as I know, this is the latest situation, for scalars. The book by Baltes and Hilf [42] is a standard account of the classic calculations in this area.

There are situations in which the Weyl conjecture is false. Gromes, [49], shows by a lattice method that for a lune of irrational angle, the conjecture is true, while it is not if the angle is a rational part of π. Bérard and Besson, [43], analyse the 2-hemisphere where the angle is π. They use a method similar to Balian and Bloch's but that is not the same in detail. Bérard, [44], considers other domains on the 2-sphere.

For a given λ, solving the coexact eigenvalues, (1), for a running l, gives

$$\begin{aligned} l_r &= \Big(\lambda + ((d-1)/2 - p)^2\Big)^{1/2} - \frac{d+1}{2} \\ &\sim -\frac{d+1}{2} + \sqrt{\lambda} + o(1) \end{aligned} \tag{64}$$

for large λ. Note that the p-dependence is all in the $o(1)$ term. This means that we can use the formula (59) without having to pick out a particular power of z.

From the definition (60), the counting function is

$$\begin{aligned} N(z,\lambda) &= G^{CE}(z,[l_r]) \\ &= G^{CE}(z, l_r - \{l_r\}), \end{aligned}$$

$[l_r]$ and $\{l_r\}$ being the integer and fractional parts of l_r, respectively, and these are functions of λ by (64).

Define $\theta(\lambda) = 2\{l_r\} - 1$, oscillating between -1 and 1, and use (59) to give the leading behaviour, for the hemisphere,

$$N_{\substack{a\\r}}(z,\lambda) \sim \frac{\lambda^{d/2}}{d!}(1+z)^{d-1} + \frac{\lambda^{(d-1)/2}}{2(d-1)!}(1+z)^{d-2}\left((1+z)(\theta \mp 1) \mp 2\right) + o(\lambda^{d/2-1}). \tag{65}$$

These formulae show that the Weyl conjecture is not valid, a result extending that of [43] to higher dimensions and to p-forms.

I have given both the absolute and relative expressions, using (48). Adding gives the *full* sphere quantity,

$$N_{\text{sphere}}(z,\lambda) \sim (1+z)^{d-1}\left(2\frac{\lambda^{d/2}}{d!} + \theta\frac{\lambda^{(d-1)/2}}{(d-1)!}\right) + o(\lambda^{d/2-1}), \tag{66}$$

and it is seen that the Weyl conjecture does not hold here either, as recognised early by Balian and Bloch [43] for scalars. Clearly condition (b) is not satisfied. Actually this will then be true for *all* the factorings, as has been shown explicitly for the hemisphere. Symmetry is responsible for this circumstance and is evidenced in the degeneracies. For a non-symmetrical domain one would expect the eigenlevels to be simple, the Weyl conjecture to be valid and periodic geodesics to be sparse.

It is useful to repeat the rather simple argument that Bérard, [44], used to show that the Weyl conjecture does not hold for tessellations of the 2-sphere. The conjecture is assumed to hold and shown to lead to a contradiction. From

(62) it is easily shown that for small ϵ and λ sufficiently large, $N(\lambda+\epsilon)-N(\lambda-\epsilon)$ is positive. This means that there has to be at least one eigenvalue in the range $\lambda+\epsilon\to\lambda-\epsilon$ but this cannot be true because, from (1),

$$\sqrt{\lambda(l)}\sim l+\frac{d+1}{2}+O(1/l),\quad \forall p,$$

and the sampling region 2ϵ can easily lie in the spectral gaps.

One sees that the second term in (66) is proportional to θ. Without, at this level, needing to go into precise detail, it is possible to replace θ by its average, namely zero, to give the second (vanishing) term in an asymptotic expansion of a, thereby, *averaged* counting function which reads

$$\overline{N}_{\text{sphere}}(z,\lambda)\sim 2\frac{\lambda^{d/2}}{d!}(1+z)^{d-1}+o(\lambda^{d/2-1}). \tag{67}$$

The absence of the second term reflects the absence of a boundary, in concordance with the Weyl conjecture, (62).

Making the same average in (65) gives

$$\overline{N}_{\substack{a\\r}}(z,\lambda)\sim\frac{\lambda^{d/2}}{d!}(1+z)^{d-1}\mp\frac{\lambda^{(d-1)/2}}{2(d-1)!}\Big((1+z)^{d-1}-2(1+z)^{d-2}\Big)+o(\lambda^{d/2-1}). \tag{68}$$

Therefore, in terms of the expression (62) for a coexact p-form,

$$B^{CE}_{\substack{a\\r}}|\partial\mathcal{M}|=\mp\frac{1}{2(d-1)!}\left[\binom{d-1}{p}-2\binom{d-2}{p}\right], \tag{69}$$

which can be compared with the general results of Blazic, Bokan and Gilkey, [50], after the general relation with the heat-kernel coefficient, $C_{1/2}$,

$$B|\partial\mathcal{M}|=\frac{C_{1/2}}{\Gamma((d+1)/2)}, \tag{70}$$

is employed. The formulae of [50] hold for the complete p-form but can be reconstructed from the coexact expressions using (13) which produces, for a p-form, from (69),

$$B|\partial\mathcal{M}|=\frac{1}{2(d-1)!}\left[\binom{d-1}{p-1}-\binom{d-1}{p}\right].$$

Hence, from (70),

$$C_{1/2}=\frac{\Gamma((d+1)/2)}{2\Gamma(d)}\left[\binom{d-1}{p-1}-\binom{d-1}{p}\right],$$

which agrees with [50] when the boundary is a $(d-1)$-sphere. This conclusion generalises Theorem 2 in Gromes, [49], to p-forms.

A more systematic averaging process, such as Riesz means, would yield the higher terms in the expansion of an averaged counting function, but one would only gain the values of the heat-kernel expansion coefficients and these can be more easily found.

11. The Polya conjecture

Because of (60) at $\lambda=\lambda_k$ and the fact that the accumulated degeneracy $G(l)\sim k$ for large λ_k, where $\lambda_k=\lambda(l)$, Weyl's asymptotic law, (61), can be turned around, for $p=0$ say,

$$\lambda_k\sim 4\pi^2\left(\frac{k}{|B_d||\mathcal{M}|}\right)^{2/d},\quad \lambda_k\to\infty, \tag{71}$$

where $|B_d|$ is the volume of the d-ball.

By contrast, the original Polya conjecture is the inequality

$$\lambda_k \geq 4\pi^2 \left(\frac{k}{|B_d||\mathcal{M}|}\right)^{2/d}, \quad k = 1, 2, \ldots \tag{72}$$

applied to *all* Dirichlet eigenvalues of the scalar Laplacian. If true, it shows that the Weyl limit is approached from above. For the Neumann case one has the opposite behaviour,

$$\mu_k \leq 4\pi^2 \left(\frac{k-1}{|B_d||\mathcal{M}|}\right)^{2/d}, \quad k = 1, 2, \ldots. \tag{73}$$

Polya proved (72) and (73) for a tiling of the 2-plane (and, essentially, of the d-plane). The conjecture is that they hold for any domain, curved or not.

The important point about the Polya conjecture is that it concerns *all* eigenvalues. Its weaker limitation to just the asymptotic behaviour is a more amenable problem and could be termed the Weyl–Polya conjecture. It is called this by Bérard, [44] corollary to Theorem 3, and I will adhere to this terminology. See also Melrose [47]. Bérard and Besson, [43], seem to use this name for the *exact* conjecture.

These authors use a counting function formulation of the conjecture, which, in the scalar case, reads $N_D(\lambda) \leq N_{as}(\lambda) \leq N_N(\lambda)$. Bérard and Besson, [43], prove this for scalars on the hemisphere, the quarter-sphere and the eighth-sphere and so it is not unreasonable to seek to extend this to p-forms and higher dimensions. I consider only the hemisphere.

The asymptotic expressions, (65) and (66), allow one to quickly check the Weyl–Polya conjecture, at least for the hemisphere. Simple algebra gives

$$N_{\substack{a\\r}}(\lambda) \sim \frac{1}{2} N_{\text{sphere}}(\lambda) \mp \frac{\lambda^{(d-1)/2}}{2(d-1)!}\Big((1+z)^{d-1} - 2(1+z)^{d-2}\Big) + o(\lambda^{d/2-1}). \tag{74}$$

For large enough λ, therefore, for each p such that $N_b(\lambda) \leq N_{*b}(\lambda)$, one clearly has

$$N_b(\lambda) \leq \frac{1}{2} N_{\text{sphere}}(\lambda) \leq N_{*b}(\lambda), \tag{75}$$

which is a version of the (asymptotic) Weyl–Polya conjecture. The form rank at which the second term in (74) changes sign is the middle one $p \sim (d-1)/2$. (See (69).) This result is the extension of the scalar inequality $N_D(\lambda) \leq N_N(\lambda)$ (which actually follows quite generally from the variational statement that the stock of Neumann functions is bigger than that of Dirichlet ones). I elaborate a little on this extension.

By analogy to the definition, (52), I introduce, the modified difference of accumulated degeneracies,

$$W(z, \sigma) = \frac{1}{1-\sigma} w(z, \sigma) = G_r^{CE}(z, \sigma) - G_a^{CE}(z, \sigma) - \frac{1}{1-\sigma} z^d.$$

This is an anti-reciprocal polynomial of degree $d-1$ in z,

$${}^*W(z, \sigma) = -W(z, \sigma),$$

a statement of duality.

Hence, writing the polynomial as

$$W(z, \sigma) = w_0 + w_1 z + \cdots + w_{d-1} z^{d-1},$$

one has $w_i = -w_{d-1-i}$ and, if d is odd, the middle coefficient $w_{(d-1)/2}$ is zero corresponding to the self-duality of the middle rank form discussed earlier.

It is then easy to show that $W(z, \sigma)$ vanishes at $z = 1$ and also, for odd d, at $z = -1$.

It is also a fact that the lower half coefficients are all negative and the upper half all positive,[5]

$$w_p < 0, \quad p = 0, 1, \ldots, d/2 - 1$$
$$w_p > 0, \quad p = d/2, 1, \ldots, d-1,$$

which constitutes our p-form generalisation of $N_D(\lambda) \leq N_N(\lambda)$.

[5] Unfortunately I have not yet been able to prove this, but symbolic manipulation verifies it.

The formula (75) is an example of a more general result. Let $\mathcal{S}_b^{\mathcal{M}}$ be any spectral quantity for the fundamental domain, $\mathcal{M}$, with boundary condition b. Then trivially, if $\mathcal{S}_b^{\mathcal{M}} \le \mathcal{S}_{*b}^{\mathcal{M}}$,

$$\mathcal{S}_b^{\mathcal{M}} \le \frac{1}{2}\mathcal{S}_p^{2\mathcal{M}} \le \mathcal{S}_{*b}^{\mathcal{M}} \tag{76}$$

where

$$\mathcal{S}_p^{2\mathcal{M}} = \mathcal{S}_b^{\mathcal{M}} + \mathcal{S}_{*b}^{\mathcal{M}} = \mathcal{S}_a^{\mathcal{M}} + \mathcal{S}_r^{\mathcal{M}}$$

is the same spectral quantity for the doubled fundamental domain, $2\mathcal{M}$, with periodic boundary conditions, p, as was introduced earlier.

Checking the exact Polya conjectures (72) and (73) is harder and I present a numerical discussion only.

The definition of the counting label, k, and the accumulated degeneracy, $G(z,l)$ or $G(p,l)$, shows that, if $\lambda_k = \lambda^{CE}(p,l)$ of (1), then

$$k_{\min}(l) \le k \le k_{\max}(l)$$

where

$$k_{\min}(l) = G_b^{CE}(p, l-1) + 1, \qquad k_{\max}(l) = G_b^{CE}(p,l), \tag{77}$$

and $k_{\min}(0) = 1$.

The conjectures (72) would then transcribe at worst to

$$\lambda^{CE}(p,l) \ge 4\pi^2 \left(\frac{k_{\min}}{|B_d||\mathcal{M}|}\right)^{2/d} = 4\pi^2 \left(\frac{G_b^{CE}(p,l-1)+1}{|B_d||\mathcal{M}|}\right)^{2/d}, \tag{78}$$

and

$$\lambda^{CE}(p,l) \le 4\pi^2 \left(\frac{k_{\max}-1}{|B_d||\mathcal{M}|}\right)^{2/d} = 4\pi^2 \left(\frac{G_{b^*}^{CE}(p,l)-1}{|B_d||\mathcal{M}|}\right)^{2/d}, \tag{79}$$

when $G_b^{CE}(p,l) \le G_{b^*}^{CE}(p,l)$.

A crude check of consistency is to take the middle rank form, i.e. $b = b^*$, when (78) and (79) imply that $\lambda^{CE}(p,l+1) > \lambda^{CE}(p,l)$.

Explicitly, on the d-hemisphere, the absolute accumulated degeneracy is

$$G_a^{CE}(p,l) = \sum_{r=0}^{d-1} \binom{l+r+1}{l}\binom{r}{p}, \tag{80}$$

with $G_{b^*}^{CE}(p,l) = G_b^{CE}(d-1-p,l)$.

Eqs. (78) and (79) read, in this particular case,

$$\lambda^{CE}(p,l) \ge (dp!(d-1-p)!(G_b^{CE}(p,l-1)+1))^{2/d}, \tag{81}$$

$$\lambda^{CE}(p,l) \le (dp!(d-1-p)!(G_{b^*}^{CE}(p,l)-1))^{2/d}. \tag{82}$$

I look only at odd d. For $p < (d-1)/2$, $b = r$ and for $p > (d-1)/2$, $b = a$ so that the relations can be tested numerically using (80). I find that the 'Dirichlet' relation, (81), is satisfied for all l, but that the 'Neumann' one, (82), is violated for the lower l values, $0, 1, \ldots,$ up to a finite number that depends on d and p.

12. Conclusion

My main formal results are the Poincaré series (47), (48), for the coexact Laplacian degeneracies on d-dimensional fundamental domains. A lot of spectral information is contained in these simple formulae. Some of the specific consequences, such as the termination of the heat-kernel expansion, are not unexpected. Results involving S^3 generally extend, in some way, to higher odd dimensions.

I note that the heat-kernel expansions are all of the conventional form. Because of the fixed points, logarithmic terms might have been expected but it seems that these are hard to generate, [51,52]. The ζ-function has the standard meromorphic structure and the implication is that the image method (the group average) automatically yields the Friedrichs extension.

It is shown that the Weyl conjecture on the asymptotic behaviour of the spectral counting function is not valid on higher dimensional tessellations, generalising a result of Bérard and Besson.

The consequences of an irrational eta invariant for the signature have yet to be determined.

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